

## GOING THROUGH A QUANTUM PHASE

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## Abstract

Phase measurements on a single-mode radiation field are examined from a system-theoretic viewpoint. Quantum estimation theory is used to establish the primacy of the Susskind-Glogower (SG) phase operator; its phase eigenkets generate the probability operator measure (POM) for maximum-likelihood phase estimation. A commuting observables description for the SG-POM on a signal  $\times$  apparatus state space is derived. It is analogous to the signal-band  $\times$  image-band formulation for optical heterodyne detection. Because heterodyning realizes the annihilation operator POM, this analogy may help realize the SG-POM. The wave function representation associated with the SG-POM is then used to prove the duality between the phase measurement and the number operator measurement, from which a number-phase uncertainty principle is obtained, via Fourier theory, without recourse to linearization. Fourier theory is also employed to establish the principle of number-ket causality, leading to a Paley-Wiener condition that must be satisfied by the phase-measurement probability density function (PDF) for a single-mode field in an arbitrary quantum state. Finally, a two-mode phase measurement is shown to afford phase-conjugate quantum communication at zero error probability with finite average photon number. Application of this construct to interferometric precision measurements is briefly discussed.

## 1. INTRODUCTION

A classical single-mode radiation field is characterized by a spatial-mode pattern  $\vec{\xi}(\vec{r})$ , an oscillation frequency  $\omega$  in rad/s, and a  $c$ -number phasor  $a$ . The latter specifies both the energy and the initial phase

shift of the field—we can take  $H = N\hbar\omega$  to be the mode energy, and  $\phi$  to be the mode phase, where

$$a = \sqrt{N}e^{i\phi}, \quad (1)$$

is the polar decomposition of  $a$ . When the single-mode classical field is quantized, its mode pattern and frequency are unchanged, but  $a$  is replaced by the annihilation operator  $\hat{a}$ . The phase problem, for this single-mode quantum field, has long been taken to mean finding a satisfactory quantum version of Eq. 1.<sup>1</sup> However, owing to the noncommutative nature of the quantum theory's operator algebra, no such decomposition exists, i.e., there is no observable  $\hat{\phi}$  such that<sup>2</sup>

$$\hat{a} = \sqrt{\hat{N}}e^{i\hat{\phi}}. \quad (2)$$

One may quibble about the order of the amplitude and phase terms on the right-side of Eq. 2, or prefer the use of  $\hat{N} + \hat{I} = \hat{a}\hat{a}^\dagger$  in lieu of  $\hat{N} = \hat{a}^\dagger\hat{a}$ , etc., but the essential issue is the nonexistence of the observable  $\hat{\phi}$ .

Until recently the Susskind-Glogower (SG) phase operator,<sup>2</sup>

$$e^{i\hat{\phi}} \equiv (\hat{a}\hat{a}^\dagger)^{-1/2}\hat{a}, \quad (3)$$

has seemed to provide the best quantum description of phase. The SG operator is non-Hermitian, and its quadratures,

$$\cos(\hat{\phi}) \equiv \text{Re}(e^{i\hat{\phi}}), \quad \text{and} \quad (4)$$

$$\sin(\hat{\phi}) \equiv \text{Im}(e^{i\hat{\phi}}), \quad (5)$$

are noncommuting observables which fail certain reasonable conditions that the cosine and sine of a phase should meet. For example, it turns out that

$$\langle \psi | \cos(\hat{\phi})^2 | \psi \rangle + \langle \psi | \sin(\hat{\phi})^2 | \psi \rangle < 1, \quad (6)$$

unless the state  $|\psi\rangle$  is orthogonal to the vacuum,  $|0\rangle$ . On the other hand, the SG-based commutator

$$[\hat{N}, \widehat{\sin(\phi)}] = i\widehat{\cos(\phi)}, \quad (7)$$

does lead to the oft-employed number phase uncertainty principle,  $\Delta N \Delta \phi \geq 1/2$ , under a high-mean-field linearization.

Lately, there has been intense renewed interest in the quantum phase problem. In what follows we will review some of the recent quantum phase work of Shapiro, Shepard, and Wong,<sup>3, 4</sup> and present some new results. Because the effort of Shapiro et al. originates from a quantum estimation theory viewpoint, that tack will be taken here as well. Because the formalism of Shapiro et al. relies on the probability operator measure (POM) description of quantum measurement—a generalization of observables not well known in the physics literature—we will begin with a brief tutorial on POM's.

## 2. POM REVIEW

The textbook approach to quantum measurement is through *observables*.<sup>5</sup> For example, consider the quadrature components of the single-mode field's annihilation operator, i.e.,

$$\hat{a}_1 \equiv \text{Re}(\hat{a}), \quad \text{and} \quad (8)$$

$$\hat{a}_2 \equiv \text{Im}(\hat{a}). \quad (9)$$

These are *continuous-spectrum* observables. In other words, they are Hermitian operators

$$\hat{a}_j^\dagger = \hat{a}_j, \quad \text{for } j = 1, 2, \quad (10)$$

with complete orthonormal (CON) eigenkets,

$$\hat{a}_j |\alpha_j\rangle_j = \alpha_j |\alpha_j\rangle_j, \quad \text{for } -\infty < \alpha_j < \infty, \quad (11)$$

$${}_j\langle\alpha'_j|\alpha_j\rangle_j = \delta(\alpha'_j - \alpha_j), \quad (12)$$

$$\hat{I} = \int_{-\infty}^{\infty} d\alpha_j |\alpha_j\rangle_j {}_j\langle\alpha_j|, \quad (13)$$

where  $\hat{I}$  is the identity operator, and  $\delta(\cdot)$  is the Dirac delta function.

Measurement of a quadrature operator, when the system is in state  $|\psi\rangle$ , gives a continuous-valued, *classical* random variable with PDF

$$p(\alpha_j | |\psi\rangle) \equiv |{}_j\langle\alpha_j|\psi\rangle|^2, \quad \text{for } -\infty < \alpha_j < \infty, j = 1, 2. \quad (14)$$

For this classical probability density to be correct, for all possible  $|\psi\rangle$ , it must satisfy

$$p(\alpha_j | |\psi\rangle) \geq 0, \quad \text{for } -\infty < \alpha_j < \infty, j = 1, 2, \quad (15)$$

and

$$\int_{-\infty}^{\infty} d\alpha_j p(\alpha_j | |\psi\rangle) = 1, \quad \text{for } j = 1, 2. \quad (16)$$

These conditions are ensured by Eq. 13, which leads to the familiar quadrature representations—essentially the position and momentum wave functions—given by

$$\begin{aligned} |\psi\rangle &= \hat{I}|\psi\rangle = \int_{-\infty}^{\infty} d\alpha_1 |\alpha_1\rangle_{11} \langle\alpha_1|\psi\rangle \\ &= \int_{-\infty}^{\infty} d\alpha_1 \psi(\alpha_1) |\alpha_1\rangle_1, \end{aligned} \quad (17)$$

and

$$\begin{aligned} |\psi\rangle &= \hat{I}|\psi\rangle = \int_{-\infty}^{\infty} d\alpha_2 |\alpha_2\rangle_{22} \langle\alpha_2|\psi\rangle \\ &= \int_{-\infty}^{\infty} d\alpha_2 \Psi(\alpha_2) |\alpha_2\rangle_2, \end{aligned} \quad (18)$$

with the obvious identifications for  $\psi(\alpha_1)$  and  $\Psi(\alpha_2)$ . Of course, the quadratures are *noncommuting* observables,

$$[\hat{a}_1, \hat{a}_2] = \frac{i}{2} \hat{I}, \quad (19)$$

so they cannot be measured *simultaneously*.<sup>5</sup>

The preceding review demonstrates that the full specification of observables, i.e., Hermitian operators with CON eigenkets, is *not* needed to produce a consistent statistical characterization of a quantum measurement. For an arbitrary quantum state, a *resolution of the identity*—an outer-product sum like Eq. 13—generates a proper classical-probability description of a quantum measurement. This is the

essence of the POM concept. Our principal purpose for introducing POM's is to accommodate measurements that are *not* observables on the state space,  $\mathcal{H}$ , of the  $\hat{a}$ -mode. The best way to introduce such nonobservable POM's is through an example. It is well known that the annihilation operator,  $\hat{a}$ , is *not* an observable—it is non-Hermitian,

$$\hat{a}^\dagger \neq \hat{a}. \quad (20)$$

Furthermore, its real and imaginary parts,  $\hat{a}_1$  and  $\hat{a}_2$ , are *noncommuting* observables— $\hat{a}$  *cannot* be measured in the usual textbook sense. However, the annihilation operator *does* have eigenkets—the coherent states,<sup>6</sup>

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad \text{for } \alpha \in \mathcal{C}, \quad (21)$$

where  $\mathcal{C}$  is the complex plane. These states are *not* orthonormal, i.e.,

$$\langle\alpha'|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha'|^2 - \frac{1}{2}|\alpha|^2 + \alpha'^*\alpha\right), \quad (22)$$

which is a consequence of the nonvanishing commutator

$$[\hat{a}, \hat{a}^\dagger] = \hat{I}, \quad (23)$$

implied by Eq. 19. Nevertheless, the coherent states are complete, in fact *overcomplete*—they form a resolution of the identity

$$\hat{I} = \frac{1}{\pi} \int_{\alpha \in \mathcal{C}} d^2\alpha |\alpha\rangle\langle\alpha|, \quad (24)$$

which defines the  $\hat{a}$ -POM. The outcome of the  $\hat{a}$ -POM is a complex-valued, continuous classical random variable with PDF

$$p(\alpha | |\psi\rangle) = \frac{1}{\pi} |\langle\alpha|\psi\rangle|^2, \quad \text{for } \alpha \in \mathcal{C}, \quad (25)$$

when the field state is  $|\psi\rangle$ . Because of Eq. 24, it follows that

$$p(\alpha | |\psi\rangle) \geq 0, \quad \text{for } \alpha \in \mathcal{C}, \quad (26)$$

and

$$\int_{\alpha \in \mathcal{C}} d^2\alpha p(\alpha | |\psi\rangle) = 1, \quad (27)$$

hold, for all  $|\psi\rangle$ .

The preceding POM has long been known.<sup>7, 8, 9</sup> It represents a measurement, in the POM sense, of the annihilation operator  $\hat{a}$ : the  $\hat{a}$ -eigenkets generate the measurement statistics, and the  $\hat{a}$ -eigenvalues are the resulting observation values. This parallels the usual observables description: the observable's eigenkets generate the measurement statistics, and the associated eigenvalues are the observation values, cf. Eq. 14.

The POM formulation is *not* in conflict with the conventional dictum that only observables can be measured. Any nonobservable POM on  $\mathcal{H}$  can be represented as a collection of commuting observables on some *larger* state space which describes the original system interacting with an appropriate apparatus.<sup>7</sup> The most familiar example of this genre is optical heterodyne detection of a single-mode signal field, which provides *both* a commuting-observables description *and* a physical realization for the  $\hat{a}$ -POM.

In optical heterodyne detection,<sup>8, 9, 10</sup> a signal field of frequency  $\nu$  is mixed with a strong local-oscillator (LO) field of frequency  $\nu - \nu_{IF}$  on the surface of a photodetector. With a unity-quantum efficiency detector, and two-channel lock-in amplification at the intermediate frequency ( $\nu_{IF}$ ), this arrangement produces a complex-valued, classical random variable,  $y$ , whose measurement statistics are identical to those of the operator<sup>9</sup>

$$\hat{y} \equiv (\hat{a}_S \otimes \hat{I}_I) + (\hat{I}_S \otimes \hat{a}_I^\dagger). \quad (28)$$

Here  $\hat{a}_j$  and  $\hat{I}_j$ , for  $j = S, I$ , are the annihilation and identity operators for the *signal* mode (frequency  $\nu$ ), and the *image* mode (frequency  $\nu - 2\nu_{IF}$ ), respectively. Both of these modes beat with the LO to produce IF waveforms, just as is the case in classical superheterodyne radio reception. It is easily verified that the real and imaginary parts of  $\hat{y}$  are *commuting* observables on the *joint* state space,  $\mathcal{H}_S \otimes \mathcal{H}_I$ , and so are simultaneously measurable in the usual sense. Ordinarily, only the signal mode carries information, i.e., the image mode is unexcited. Under these circumstances, the PDF for the observed  $y$ -value

reduces to<sup>8, 9</sup>

$$p(y | |\psi\rangle_S) = \frac{1}{\pi} |{}_S\langle y | \psi \rangle_S|^2, \quad \text{for } y \in \mathcal{C}, \quad (29)$$

where  $|y\rangle_S$  is the signal-mode coherent state with eigenvalue  $y$ , and  $|\psi\rangle_S$  is an arbitrary signal-mode state. Comparison of Eqs. 29 and 25 completes the demonstration that heterodyning—a pair of commuting observables on an extended state space with an unexcited image mode—realizes the  $\hat{a}_S$ -POM.

### 3. PHASE ESTIMATION

Rather than seeking a quantum formalism for phase by pursuing a quantum version of Eq. 1, Shapiro, Shepard, and Wong<sup>3, 4</sup> approached the problem from the estimation theory viewpoint. Consider the following abstract quantum estimation problem. A single-mode input field of annihilation operator  $\hat{a}_{IN}$  and quantum state  $|\psi\rangle_{IN}$  undergoes an unknown, non-random, c-number phase shift  $\Phi$ , yielding a single-mode output field of annihilation operator

$$\hat{a} = e^{i\Phi} \hat{a}_{IN}, \quad (30)$$

in state

$$|\psi\rangle = \exp(i\Phi \hat{a}_{IN}^\dagger \hat{a}_{IN}) |\psi\rangle_{IN}. \quad (31)$$

By making an appropriate quantum measurement on the  $\hat{a}$  mode, and knowing the input state  $|\psi\rangle_{IN}$ , we are to estimate the phase shift  $\Phi$ . An interferometric phase measurement can be embedded into this scheme by placing appropriate constraints on the allowable quantum measurement. Optimizing a phase measurement within this more restricted environment *cannot* outperform the behavior obtained from an unfettered measurement optimization. Indeed, we should expect that *joint* optimization of the quantum measurement *and* the input state will yield superior phase estimation performance.

Without loss of generality, we can confine the phase shift to a  $2\pi$ -rad interval, i.e., we can assume that  $-\pi < \Phi \leq \pi$ . The class of POM's we must optimize over, in order to find the best phase measurement,

can be taken to be  $\{d\hat{\Pi}(\phi) : -\pi < \phi \leq \pi\}$ , where

$$d\hat{\Pi}(\phi) = d\hat{\Pi}(\phi)^\dagger, \quad (32)$$

and

$$\hat{I} = \int_{-\pi}^{\pi} d\hat{\Pi}(\phi), \quad (33)$$

on the state space of the output mode,  $\hat{a}$ . The conditional probability density, given  $\Phi$ , for obtaining a phase value  $\phi$  from this POM is

$$p(\phi | \Phi) = \frac{\langle \psi | d\hat{\Pi}(\phi) | \psi \rangle}{d\phi}, \quad \text{for } -\pi < \phi, \Phi \leq \pi, \quad (34)$$

where  $|\psi\rangle$  is the state of the  $\hat{a}$ -mode.

In *classical* estimation theory, the maximum-likelihood (ML) estimate  $\Phi_{ML}$  of an unknown, nonrandom, phase shift  $\Phi$ , based on a noisy phase-shift observation  $\phi$ , of known PDF  $p(\phi | \Phi)$ , is the phase shift which maximizes the likelihood of getting the observed datum, i.e.,

$$\Phi_{ML}(\phi) = \arg \max_{-\pi < \theta \leq \pi} p(\phi | \theta). \quad (35)$$

Often, the ML phase estimate equals the observed phase shift, because  $p(\phi | \theta)$  has its peak at  $\theta = \phi$ , for  $-\pi < \phi \leq \pi$ . Such is the case for phase estimation in additive white Gaussian noise.<sup>11</sup> It then follows the  $p(\phi | \phi)$ , the *peak* likelihood, is a simple, but meaningful, performance measure for  $\Phi_{ML}$ . Indeed, its reciprocal,

$$\delta\phi \equiv \frac{1}{p(\phi | \phi)}, \quad (36)$$

is the PDF's width for the case of a uniform distribution; if the distribution is Gaussian, then we have  $\delta\phi = \sqrt{2\pi}\Delta\phi$ , where  $\Delta\phi$  is the root-mean-square (RMS) error.

Our problem is one of *quantum* estimation theory, namely, choosing the POM,  $d\hat{\Pi}(\phi)$ , and the input state,  $|\psi\rangle_{IN}$ , to optimize our estimate of the phase shift  $\Phi$ . For a *given* POM *and* input state, Eq. 34 supplies the PDF needed to perform classical ML estimation. In this quantum setting, however, the observed phase value  $\phi$  is, by presumption, our estimate of  $\Phi$ . Thus, in

order for this estimate to be one of maximum likelihood, we can restrict our attention to POM's satisfying

$$\phi = \arg \max_{-\pi < \theta \leq \pi} p(\phi | \theta),$$

$$\text{for } -\pi < \phi \leq \pi, \quad (37)$$

and optimize our estimate over  $d\hat{\Pi}$  and  $|\psi\rangle_{IN}$  by maximizing the peak likelihood—minimizing  $\delta\phi$ —averaged over all possible  $\Phi$  values. Here it is known that, for the input state whose number representation,  $\psi_n \equiv \langle n | \psi \rangle$ , is

$$\psi_n = |\psi_n| e^{i\chi_n}, \quad \text{for } n = 0, 1, 2, \dots, \quad (38)$$

$\delta\phi$  is minimized by the following POM,<sup>7</sup>

$$d\hat{\Pi}(\phi) = |e^{i\phi}, \psi\rangle \langle e^{i\phi}, \psi| \frac{d\phi}{2\pi},$$

$$\text{for } -\pi < \phi \leq \pi, \quad (39)$$

where

$$|e^{i\phi}, \psi\rangle \equiv \sum_{n=0}^{\infty} e^{i(n\phi + \chi_n)} |n\rangle. \quad (40)$$

Moreover, the reciprocal peak-likelihood that results when we use this optimum POM to estimate  $\Phi$  is easily shown to be

$$\delta\phi = 2\pi \left| \langle e^{i\phi}, \psi | \psi \rangle \right|^{-2}$$

$$= 2\pi \left( \sum_{n=0}^{\infty} |\psi_n| \right)^{-2}, \quad (41)$$

which is *independent* of the phases  $\{\chi_n\}$ . In fact,  $p(\phi | \Phi)$  is independent of the  $\{\chi_n\}$ .

We can exploit the  $\{\chi_n\}$  independence to good purposes by assuming, without loss of generality, that the input state has positive real  $\psi_n$ . Equation 40 then reduces to

$$|e^{i\phi}, \psi\rangle = |e^{i\phi}\rangle \equiv \sum_{n=0}^{\infty} e^{in\phi} |n\rangle,$$

$$\text{for } -\pi < \phi \leq \pi, \quad (42)$$

which is the number-ket expansion of the SG phase operator's (infinite-energy) eigenkets, viz.

$$\widehat{e^{i\phi}} |e^{i\phi}\rangle = e^{i\phi} |e^{i\phi}\rangle,$$

$$\text{for } -\pi < \phi \leq \pi. \quad (43)$$

This says that the SG-POM is the quantum measurement for ML phase estimation in the general measurement configuration when the input state has a positive real number representation. In other words, the phase eigenkets of the SG operator generate the resolution of the identity,

$$\hat{I} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi |e^{i\phi}\rangle \langle e^{i\phi}|, \quad (44)$$

needed for ML quantum phase estimation in this case. For an *arbitrary* input state, the optimum POM from Eq. 39 is equivalent to performing the unitary state transformation

$$\hat{U} \equiv \sum_{n=0}^{\infty} e^{-i\chi_n} |n\rangle \langle n|, \quad (45)$$

followed by the SG-POM.

To achieve the goal of *jointly* optimizing phase-estimation performance over *both* the measurement *and* the input state, it only remains for us to minimize  $\delta\phi$ , from Eq. 41, by appropriate choice of  $|\psi\rangle_{IN}$ . This problem has been addressed,<sup>3, 4</sup> and the state

$$\psi_n = \frac{A}{1+n},$$

$$\text{for } n = 0, 1, 2, \dots, M < \infty, \quad (46)$$

where  $A$  is a normalization constant and  $M$  is a truncation parameter, has been shown to achieve

$$\delta\phi \sim 1/N^2, \quad (47)$$

in terms of its average photon number,  $N = \langle \hat{a}^\dagger \hat{a} \rangle$ . This performance is far superior to the  $\delta\phi \sim 1/N$  reciprocal peak likelihood capability of optimized squeezed-state interferometry. However, the phase measurement PDF for the Eq. 46 state is a heavy-tailed distribution, viz., its RMS phase error,  $\Delta\phi$ , is essentially independent of  $N$ . Thus, the degree to which this reciprocal peak likelihood advantage can be usefully exploited has yet to be established.<sup>12</sup> In what follows, therefore, we will concentrate on the SG-POM, in that it constitutes the maximum-likelihood quantum phase measurement for *all* quantum states.

#### 4. SG POM

The Susskind-Glogower operator,<sup>2, 13</sup>

$$\widehat{e^{i\phi}} \equiv (\hat{a}\hat{a}^\dagger)^{-1/2}\hat{a}, \quad (48)$$

affords a well-defined polar decomposition of  $\hat{a}$ ,

$$\hat{a} = \sqrt{\hat{N} + \hat{I}} \widehat{e^{i\phi}}, \quad (49)$$

in terms of energy (number) and phase operators. Using the number-ket expansions of  $\hat{N}$  and  $\hat{a}$  we have that

$$\widehat{e^{i\phi}} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|, \quad (50)$$

from which it follows that

$$\begin{aligned} \widehat{e^{-i\phi}} &\equiv \widehat{e^{i\phi}}^\dagger \\ &= \sum_{n=0}^{\infty} |n+1\rangle\langle n| \neq \widehat{e^{i\phi}}, \end{aligned} \quad (51)$$

and

$$[\widehat{e^{i\phi}}, \widehat{e^{-i\phi}}] = |0\rangle\langle 0|. \quad (52)$$

In words, the SG phase operator is *not* Hermitian, and does *not* commute with its adjoint. Thus, as was seen earlier for  $\hat{a}$  itself, the quadrature components of the SG operator,

$$\widehat{\cos(\phi)} \equiv \text{Re}(\widehat{e^{i\phi}}), \quad \text{and} \quad (53)$$

$$\widehat{\sin(\phi)} \equiv \text{Im}(\widehat{e^{i\phi}}), \quad (54)$$

are *noncommuting* observables,

$$[\widehat{\cos(\phi)}, \widehat{\sin(\phi)}] = \frac{i}{2}|0\rangle\langle 0|. \quad (55)$$

The SG-POM derives from the fact that  $\widehat{e^{i\phi}}$  has an *overcomplete* set of eigenkets, cf. the Sect. 2 discussion of the  $\hat{a}$ -POM. By direct substitution of Eq. 50, we can verify that

$$\begin{aligned} \widehat{e^{i\phi}}|e^{i\phi}\rangle &= |e^{i\phi}\rangle, \\ &\text{for } -\pi < \phi \leq \pi, \end{aligned} \quad (56)$$

where  $|e^{i\phi}\rangle$  has the number-ket representation given in Eq. 42. That these kets

resolve the identity is also easily shown,

$$\begin{aligned} &\int_{-\pi}^{\pi} d\phi |e^{i\phi}\rangle\langle e^{i\phi}| \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{-\pi}^{\pi} d\phi e^{i(n-m)\phi} |n\rangle\langle m| \\ &= 2\pi \sum_{n=0}^{\infty} |n\rangle\langle n| = 2\pi \hat{I}. \end{aligned} \quad (57)$$

That they are *not* orthonormal can be demonstrated from some simple Fourier transform manipulations,<sup>14</sup>

$$\begin{aligned} \langle e^{i\phi'} | e^{i\phi} \rangle &= \sum_{n=0}^{\infty} e^{-in(\phi' - \phi)} \\ &= \frac{1}{2} \left( \sum_{n=-\infty}^{\infty} e^{-in(\phi' - \phi)} \right. \\ &\quad \left. + \sum_{n=-\infty}^{\infty} \text{sgn}(n) e^{-in(\phi' - \phi)} + 1 \right) \\ &= \pi \delta(\phi' - \phi) \\ &\quad - \frac{i}{2} \cot\left(\frac{\phi' - \phi}{2}\right) + \frac{1}{2} \end{aligned} \quad (58)$$

Here,  $\delta(\cdot)$  is the Dirac delta function, and

$$\text{sgn}(n) \equiv \begin{cases} -1, & n < 0, \\ 0, & n = 0, \\ 1, & n > 0, \end{cases} \quad (59)$$

is the signum function.

#### 5. COMMUTING OBSERVABLES

Recall that a POM on  $\mathcal{H}$  which is *not* an observable on that space can be represented as a collection of commuting observables on a larger, signal  $\times$  apparatus state space,  $\mathcal{H} \otimes \mathcal{H}_A$ , with the apparatus placed in some appropriate state. We now develop such a representation for the SG-POM. Aside from alleviating the qualms of those who believe only in observables, this representation may guide us to a realization of the SG-POM—the commuting-observables description of the  $\hat{a}$ -POM is intimately connected with its heterodyne-detection realization.

Let  $\hat{a}_A$  be the annihilation operator of an apparatus mode, whose state space,

$\mathcal{H}_A$ , is spanned by its number kets,  $\{|n\rangle_A : n = 0, 1, 2, \dots\}$ . The non-Hermitian operator

$$\hat{Y} \equiv \left( e^{i\phi} \otimes |0\rangle_A \langle 0| \right) + \left( |0\rangle \langle 0| \otimes e^{-i\phi} \right), \quad (60)$$

where

$$e^{-i\phi} \equiv \hat{a}_A^\dagger (\hat{a}_A \hat{a}_A^\dagger)^{-1/2}, \quad (61)$$

is easily shown to commute with its adjoint. Here,  $\hat{Y}$  is an operator on the joint state space  $\mathcal{H} \otimes \mathcal{H}_A$ , and  $e^{i\phi} \hat{a}_A$  is the apparatus mode's SG phase operator.

Because  $[\hat{Y}, \hat{Y}^\dagger] = 0$ , the quadrature components of  $\hat{Y}$ —denoted  $\hat{Y}_1$  and  $\hat{Y}_2$ —are commuting observables, which can be measured *simultaneously*, i.e.,  $\hat{Y} = \hat{Y}_1 + \hat{Y}_2$  can be measured in the usual sense. Solving for the eigenkets and eigenvalues of  $\hat{Y}$  we find that signal  $\times$  apparatus number ket

$$|Y\rangle \equiv |n\rangle |m\rangle_A, \quad \text{for } nm > 0, \quad (62)$$

is a  $\hat{Y}$ -eigenket with zero eigenvalue, and

$$|Y\rangle \equiv \frac{1}{\sqrt{2\pi}} (|0\rangle |0\rangle_A + \sum_{n=1}^{\infty} (e^{in\phi} |n\rangle |0\rangle_A + e^{-in\phi} |0\rangle |n\rangle_A)), \quad \text{for } -\pi < \phi \leq \pi, \quad (63)$$

is a  $\hat{Y}$ -eigenket whose associated eigenvalue is  $e^{i\phi}$ . Collectively, these comprise a CON set from which we have that measurement of  $\hat{Y}$ , when the signal  $\times$  apparatus state is  $|\psi\rangle_{S \times A} \in \mathcal{H} \otimes \mathcal{H}_A$ , yields a *mixed* classical random variable,  $Y$ , which takes on *either* the *discrete* value 0, *or* a value from the *continuum*  $\{e^{i\phi} : -\pi < \phi \leq \pi\}$ . The former occurs with discrete probability

$$\Pr(0 | |\psi\rangle_{S \times A}) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\psi_{nm}|^2; \quad (64)$$

the probability density for the latter is

$$p(\phi | |\psi\rangle_{S \times A}) = \frac{1}{2\pi} |\psi_{00} + \sum_{n=1}^{\infty} (e^{-in\phi} \psi_{n0} + e^{in\phi} \psi_{0n})|^2, \quad \text{for } -\pi < \phi \leq \pi, \quad (65)$$

where  $\psi_{nm} \equiv {}_A\langle m | \langle n | \psi_{S \times A} \rangle$ . These two are properly normalized in that for *all*  $|\psi\rangle_{S \times A}$  we have that

$$\Pr(0 | |\psi\rangle_{S \times A}) + \int_{-\pi}^{\pi} d\phi p(\phi | |\psi\rangle_{S \times A}) = 1, \quad (66)$$

as required by classical probability theory.

Now, the commuting-observables representation of the SG-POM is at hand. Suppose we measure  $\hat{Y}$  when the apparatus mode is unexcited, i.e.,  $|\psi\rangle_{S \times A} = |\psi\rangle |0\rangle_A$ , where  $|\psi\rangle \in \mathcal{H}$  is an *arbitrary* signal-mode state and  $|0\rangle_A$  is the apparatus mode's vacuum state. Then the discrete value zero is *never* obtained, and the PDF for obtaining  $Y = e^{i\phi}$  reduces to

$$\begin{aligned} p(\phi | |\psi\rangle |0\rangle_A) &= \frac{1}{2\pi} \left| \sum_{n=0}^{\infty} e^{-in\phi} \psi_n \right|^2 \\ &= \frac{1}{2\pi} |\langle e^{i\phi} | \psi \rangle|^2, \end{aligned} \quad \text{for } -\pi < \phi \leq \pi, \quad (67)$$

realizing the SG-POM statistics for an *arbitrary* state of the  $\hat{a}$ -mode.

The equivalence of the SG-POM to the  $\hat{Y}$  measurement with an unexcited apparatus mode allows us to clarify some basic points. First, because the SG operator does not commute with its adjoint, it is really the operator analog of the *c*-number  $e^{i\phi}$  from the classical single-mode field. In other words, there is *no* Hermitian phase operator,  $\hat{\phi}$ , on  $\mathcal{H}$  such that  $\exp(i\hat{\phi}) = (\hat{a}\hat{a}^\dagger)^{-1/2}\hat{a}$ . Restated in terms of the quadratures of the SG operator, this means that  $\cos(\phi) \neq \cos(\hat{\phi})$ , and  $\sin(\phi) \neq \sin(\hat{\phi})$ . As a result, the classical trigonometric identity,

$$\cos(\phi)^2 + \sin(\phi)^2 = 1, \quad \text{for } -\pi < \phi \leq \pi, \quad (68)$$

does *not* apply to the quadratures of the SG phase operator, e.g., because

$$\widehat{\cos(\phi)}^2 + \widehat{\sin(\phi)}^2 = \hat{I} - \frac{|0\rangle \langle 0|}{2}, \quad (69)$$

any  $|\psi\rangle$  with  $\psi_0 \neq 0$  gives

$$\langle \psi | \widehat{\cos(\phi)}^2 | \psi \rangle + \langle \psi | \widehat{\sin(\phi)}^2 | \psi \rangle < 1. \quad (70)$$

However, the outcome of the SG-POM is a phasor  $e^{i\phi}$ —the  $\hat{Y}$  measurement with a vacuum-state apparatus mode yields a complex-valued, continuous classical random variable  $Y = e^{i\phi}$ , where  $-\pi < \phi \leq \pi$ . Thus, we have that  $|Y|^2 = 1$ , *with probability one*.

The second point to note regarding the SG-POM is the relation of its mean value to those of the SG quadratures. Using

$$\begin{aligned} \hat{Y}_1 &= (\widehat{\cos(\phi)} \otimes |0\rangle_{AA} \langle 0|) \\ &+ (|0\rangle \langle 0| \otimes \widehat{\cos(\phi)}_A), \end{aligned} \quad (71)$$

and

$$\begin{aligned} \hat{Y}_2 &= (\widehat{\sin(\phi)} \otimes |0\rangle_{AA} \langle 0|) \\ &- (|0\rangle \langle 0| \otimes \widehat{\sin(\phi)}_A), \end{aligned} \quad (72)$$

and assuming an unexcited apparatus mode, we find that

$${}_A \langle 0 | \langle \psi | \hat{Y}_1 | \psi \rangle | 0 \rangle_A = \langle \psi | \widehat{\cos(\phi)} | \psi \rangle, \quad (73)$$

and

$${}_A \langle 0 | \langle \psi | \hat{Y}_2 | \psi \rangle | 0 \rangle_A = \langle \psi | \widehat{\sin(\phi)} | \psi \rangle, \quad (74)$$

for all  $\hat{a}$ -mode states,  $|\psi\rangle$ . What this says is that *averages* of the *classical*  $\cos(\phi)$  and  $\sin(\phi)$  random variables obtained from the SG-POM coincide with *averages* of the SG phase operator's quadratures. To the extent that the quadrature mean values comprise the information of interest, we can conclude that the SG-POM provides a proper quantum measurement description for simultaneous extraction of this information from both quadratures.

## 6. UNCERTAINTY PRINCIPLE

Number-ket expansions of the quadrature operators  $\widehat{\cos(\phi)}$  and  $\widehat{\sin(\phi)}$  lead to the commutator

$$[\hat{N}, \widehat{\sin(\phi)}] = i\widehat{\cos(\phi)}, \quad (75)$$

and the associated uncertainty principle

$$\langle \Delta \hat{N}^2 \rangle \langle \Delta \widehat{\sin(\phi)}^2 \rangle \geq \frac{1}{4} |\langle \widehat{\cos(\phi)} \rangle|^2. \quad (76)$$

Equation 76 is valid for arbitrary states, but its utility, in this general form, is somewhat limited. First, the minimum uncertainty product is state dependent—a consequence of Eq. 75 not being a  $c$ -number commutator. Second, the principle does not directly address the variance of a *phase* measurement—it is the  $\widehat{\sin(\phi)}$  operator whose variance appears.

It is common practice to use the linearized form of Eq. 76,

$$\Delta N \Delta \phi \geq \frac{1}{2}, \quad (77)$$

which applies for states meeting the high-mean-field condition,

$$\langle \hat{N} \rangle \approx |\langle \hat{a} \rangle|^2 \gg 1. \quad (78)$$

The linearized result, while useful, can be abused. Number kets have zero number-measurement uncertainty, and  $\langle \hat{a} \rangle = 0$ ,  $\langle \widehat{\cos(\phi)} \rangle = 0$ , hence the general result leads to the correct number-ket limit,

$$\langle \Delta \hat{N}^2 \rangle \langle \Delta \widehat{\sin(\phi)}^2 \rangle \geq 0, \quad (79)$$

whereas the linearized form is inapplicable.

Although the SG-POM does *not* alleviate the state-dependent nature of number-phase uncertainty limits, it *does* lead to an uncertainty principle which directly addresses phase variance. Our route to this principle—through Fourier theory—has the following motivation. The time-bandwidth uncertainty principle for the continuous-time Fourier transform (CTFT)<sup>14</sup> can be applied to the normalized position and momentum wave functions,  $\psi(\alpha_1)$ , and  $\Psi(\alpha_2)$ , because they satisfy the Fourier transform relations<sup>5</sup>

$$\begin{aligned} \Psi(\alpha_2) &= \int_{-\infty}^{\infty} \frac{d\alpha_1}{\sqrt{\pi}} \psi(\alpha_1) e^{-i2\alpha_1\alpha_2}, \end{aligned} \quad (80)$$

and

$$\begin{aligned} \psi(\alpha_1) &= \int_{-\infty}^{\infty} \frac{d\alpha_2}{\sqrt{\pi}} \Psi(\alpha_2) e^{i2\alpha_1\alpha_2}. \end{aligned} \quad (81)$$



The result of this procedure can be reduced to

$$\langle \Delta \hat{a}_1^2 \rangle \langle \Delta \hat{a}_2^2 \rangle \geq \frac{1}{16}, \quad (82)$$

which is the Heisenberg uncertainty principle for the annihilation operator's quadratures.

Because of Eq. 44, any state  $|\psi\rangle$  has a phase representation

$$\Psi(e^{i\phi}) \equiv \langle e^{i\phi} | \psi \rangle, \quad \text{for } -\pi < \phi \leq \pi, \quad (83)$$

such that

$$\begin{aligned} |\psi\rangle &= \hat{I}|\psi\rangle \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \Psi(e^{i\phi}) |e^{i\phi}\rangle. \end{aligned} \quad (84)$$

The phase representation of  $|\psi\rangle$  is intimately related to its number-ket representation,  $\psi_n \equiv \langle n | \psi \rangle$ —they are a Fourier transform pair

$$\Psi(e^{i\phi}) = \sum_{n=0}^{\infty} \psi_n e^{-in\phi} \quad (85)$$

and

$$\psi_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \Psi(e^{i\phi}) e^{in\phi}, \quad (86)$$

as can be seen from Eqs. 83 and 42. In other words,  $\Psi(e^{i\phi})$  and  $\psi_n$  constitute phase and number *wave functions*, which are capable of representing arbitrary states. The complementarity of the number operator measurement—whose probability distribution is  $\text{Pr}(\hat{N} = n) = |\psi_n|^2$ —and the SG POM—whose probability density function is  $p(\phi) = |\Psi(e^{i\phi})|^2/2\pi$ —then follows from the Fourier relations, Eq. 85 and 86. Thus, to obtain a number-phase uncertainty principle for the product of the number-operator variance and the SG-POM variance, we shall exploit this complementarity by paralleling the standard Fourier derivation of Eq. 82.

With  $\langle \Delta \hat{N}^2 \rangle$  denoting the number-measurement variance and  $\langle \Delta \phi^2 \rangle$  the SG-POM variance, when the field is in an *arbitrary* state  $|\psi\rangle$ , we have that

$$\langle \Delta \hat{N}^2 \rangle \langle \Delta \phi^2 \rangle$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} (n - \bar{n})^2 |\psi_n|^2 \\ &\times \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} (\phi - \bar{\phi})^2 |\Psi(e^{i\phi})|^2 \end{aligned} \quad (87)$$

$$\begin{aligned} &= \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \left| \frac{d\Psi'(e^{i\phi})}{d\phi} \right|^2 \\ &\times \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} (\phi - \bar{\phi})^2 |\Psi'(e^{i\phi})|^2 \end{aligned} \quad (88)$$

$$\begin{aligned} &\geq \left| \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} (\phi - \bar{\phi}) \Psi'(e^{i\phi})^* \right. \\ &\times \left. \frac{d\Psi'(e^{i\phi})}{d\phi} \right|^2 \end{aligned} \quad (89)$$

$$\begin{aligned} &\geq \left\{ \text{Re} \left[ \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} (\phi - \bar{\phi}) \Psi'(e^{i\phi})^* \right. \right. \\ &\times \left. \left. \frac{d\Psi'(e^{i\phi})}{d\phi} \right] \right\}^2 \end{aligned} \quad (90)$$

$$= \frac{1}{2\pi} [p(\pi | |\psi\rangle) - 1]^2. \quad (91)$$

In this development:  $\bar{n}$  and  $\bar{\phi}$  are the mean values of the number and SG-POM measurements, respectively, on the state  $|\psi\rangle$ ;  $\Psi'(e^{i\phi}) \equiv \Psi(e^{i\phi})e^{i\bar{n}\phi}$ ; the Schwarz inequality has been used in Eq. 89; and the integration necessary to obtain Eq. 91 follows from the SG-POM's PDF,  $p(\phi | |\psi\rangle) = |\Psi(e^{i\phi})|^2/2\pi$ .

Unlike the usual number-phase uncertainty principle, i.e., Eq. 76, our result does *not* require any linearization before it can be applied to phase variance. Equation 91 is *still* state dependent, but this is unavoidable. When  $|\psi\rangle$  is a number state, we have

$$p(\phi | |n\rangle) = \frac{1}{2\pi}, \quad \text{for } -\pi < \phi \leq \pi, \quad (92)$$

a uniform distribution, which is maximally random, but still has finite variance. There is no contradiction with Eq. 91 in this case, even though  $\Delta N = 0$  for a number ket; the uniform PDF causes the right member of Eq. 91 to vanish. On the other hand, when  $|\psi\rangle$  is a high-mean-field state, we will have  $p(\pi | |\psi\rangle) \ll 1$ , so that Eq. 91 reproduces the standard linearized formula, Eq. 77. Indeed, for *any* state satisfying  $p(\pi | |\psi\rangle) \ll 1$ , we have that Eq. 77 holds. This makes the SG-POM

derivation of Eq. 77 more robust than linearization of Eq. 76.

## 7. Number-Ket Causality

The SG-POM underlies maximum-likelihood quantum phase measurement for *all* quantum states. Given the problems associated with minimizing the SG-POM's reciprocal peak-likelihood  $\delta\phi$ , by choice of input state,<sup>3, 4, 12</sup> a different state-selection criterion may be worth considering. In this vein, it is germane to ask the following question. What SG-POM phase PDF's can be realized by choice of input state  $|\psi\rangle$ ? It turns out that linear system theory has the answer.

The Susskind-Glogower probability operator measurement on a state  $|\psi\rangle$  results in a classical random variable  $\phi$  with probability density function

$$p(\phi | |\psi\rangle) = \frac{|\Psi(e^{i\phi})|^2}{2\pi}, \quad \text{for } -\pi < \phi \leq \pi. \quad (93)$$

Here,  $\Psi(e^{i\phi})$  is the phase representation of the state  $|\psi\rangle$ . According to Eq. 86, the phase representation is the Fourier transform of the number representation. The latter is a *one-sided*, discrete-parameter sequence that is the inverse Fourier transform of  $\Psi(e^{i\phi})$ , i.e.,

$$\begin{aligned} \psi_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \Psi(e^{i\phi}) e^{in\phi} \\ &= \begin{cases} \langle n | \psi \rangle, & \text{for integer } n \geq 0, \\ 0, & \text{for integer } n < 0. \end{cases} \quad (94) \end{aligned}$$

In system-theory parlance, the Fourier pair  $\psi_n \longleftrightarrow \Psi(e^{i\phi})$  is analogous to that for a discrete-time waveform on an unbounded interval and its periodic, continuous-frequency Fourier transform, i.e., the discrete-time Fourier transform (DTFT).<sup>14</sup> More importantly, saying that  $\psi_n$  is one-sided is equivalent to saying that a discrete-time waveform is *causal*, viz. it could be the impulse response of a causal, linear time-invariant system. Determining what  $p(\phi | |\psi\rangle)$  are possible from Eq. 93 is then the same as determining what  $|\Psi(e^{i\phi})|$  are Fourier-transform magnitudes of one-sided  $\{\psi_n\}$ . To emphasize

the connection with causal waveforms, we introduce the term *number-ket causality* for the condition Eq. 94. This is a well-studied problem in linear systems, so results are immediately available.<sup>14, 15</sup>

From Eq. 93 and the Paley-Wiener theorem<sup>15</sup> we have that  $p(\phi | |\psi\rangle)$  must satisfy

$$\int_{-\pi}^{\pi} d\phi |\ln[p(\phi | |\psi\rangle)]| < \infty, \quad (95)$$

for all number-ket causal  $\Psi(e^{i\phi})$ . From this condition, it follows that *no* state can confine the phase-measurement PDF to a subinterval of  $(-\pi, \pi]$ , e.g., the uniform density,

$$\begin{aligned} p(\phi | |\psi\rangle) &= \frac{1}{\delta\phi}, \\ \text{for } |\phi| &\leq \delta\phi/2 < \pi, \end{aligned} \quad (96)$$

is *impossible*. The Paley-Wiener condition is *both* necessary *and* sufficient, i.e., if a PDF obeys Eq. 95, then there is a state which gives this density through Eq. 93. Indeed, there are an *infinite* number of such states, because Eq. 93 only constrains the *magnitude* of the phase representation. One such state can be obtained explicitly via the discrete-parameter Hilbert transform. The procedure is as follows. For a PDF obeying Eq. 95, set

$$\begin{aligned} |\Psi(e^{i\phi})| &= \sqrt{2\pi p(\phi | |\psi\rangle)}, \\ \text{for } -\pi &< \phi \leq \pi. \end{aligned} \quad (97)$$

Next, find the discrete-parameter Hilbert transform,<sup>14</sup>

$$\begin{aligned} \arg[\Psi(e^{i\phi})] &= \mathcal{P} \int_{-\pi}^{\pi} \frac{d\phi'}{2\pi} \ln[|\Psi(e^{i\phi'})|] \\ &\quad \times \cot\left(\frac{\phi' - \phi}{2}\right), \\ \text{for } -\pi &< \phi \leq \pi, \end{aligned} \quad (98)$$

where  $\mathcal{P}$  denotes Cauchy principal value. Equations 97 and 98 then comprise the magnitude and phase—the polar form—of a properly-normalized phase representation  $\Psi(e^{i\phi})$  with the prescribed phase-measurement statistics *and* a number-ket causal inverse Fourier transform.

The preceding phase representation construction for a state with prescribed SG-POM statistics is by no means unique. Equation 97 constrains  $|\Psi(e^{i\phi})|$ , but no restriction is placed on  $\arg[\Psi(e^{i\phi})]$ . Consider a number-ket causal function,  $\{h_n : n = 0, 1, 2, \dots\}$ , whose Fourier transform has unity magnitude, viz.

$$H(e^{i\phi}) \equiv \sum_{n=0}^{\infty} h_n e^{-in\phi}, \quad \text{for } -\pi < \phi \leq \pi, \quad (99)$$

obeys

$$|H(e^{i\phi})| = 1, \quad \text{for } -\pi < \phi \leq \pi. \quad (100)$$

Such a function is known in digital-filter theory as an all-pass filter; were  $\{h_n : n = 0, 1, 2, \dots\}$  the impulse response of a discrete-time, linear, time-invariant filter, the associated frequency response would pass all frequencies with neither attenuation nor gain. The prototypical example of an all-pass filter is obtained—in the  $z$ -transform domain—by balancing  $H(z)$ -poles within the unit circle with  $H(z)$ -zeros outside the unit circle to achieve<sup>14</sup>

$$H(e^{i\phi}) = \prod_{k=1}^K \frac{e^{-i\phi} - p_k^*}{1 - p_k e^{-i\phi}}, \quad \text{for } -\pi < \phi \leq \pi, \quad \text{where } |p_k| < 1, \quad \text{for } k = 1, 2, \dots, K. \quad (101)$$

Now, suppose we assemble the phase representation

$$\Psi'(e^{i\phi}) \equiv \Psi(e^{i\phi})H(e^{i\phi}), \quad \text{for } -\pi < \phi \leq \pi, \quad (102)$$

where  $\Psi(e^{i\phi})$  is constructed according to Eqs. 97 and 98 for a desired phase-measurement PDF, and  $H(e^{i\phi})$  is an all-pass phase representation from Eq. 101. The convolution-multiplication theorem of Fourier analysis, plus the fact that convolving two causal functions produces a causal function,<sup>14</sup> guarantees that  $\Psi'(e^{i\phi})$  is a properly normalized, number-ket causal phase representation; the all-pass nature of  $H(e^{i\phi})$  implies that  $\Psi'(e^{i\phi})$  has the desired SG-POM statistics. Because this process holds for all  $K \geq 1$  and for all pole locations within the unit circle,

there is an uncountable infinity of states which have the same SG-POM statistics. Nevertheless, the state constructed via the discrete-parameter Hilbert transform has a unique advantage—it is the minimum average photon-number state with the prescribed phase-measurement PDF.

The proof of the minimum average photon-number property follows almost immediately from available linear-system results. Let  $\{\Psi(e^{i\phi}) : -\pi < \phi \leq \pi\}$  and  $\{\psi_n : n = 0, 1, 2, \dots\}$  be the phase and number representations of the state  $|\psi\rangle$ , obtained via Eqs. 97, 98, and 94, that realizes a particular phase PDF. Similarly, let  $\{\Psi'(e^{i\phi}) : -\pi < \phi \leq \pi\}$  and  $\{\psi'_n : n = 0, 1, 2, \dots\}$  be the phase and number representations of any *other* state,  $|\psi'\rangle$ , with the *same* SG-POM statistics. Then, we have that<sup>14</sup>

$$\sum_{n=0}^{M-1} (|\psi_n|^2 - |\psi'_n|^2) \geq 0, \quad \text{for } M = 1, 2, 3, \dots \quad (103)$$

Physically, this says that, of all states with the desired phase behavior, the Hilbert-transform generated state concentrates its number-ket content closest to the vacuum. Because both  $|\psi\rangle$  and  $|\psi'\rangle$  are normalized, i.e., unit-length, states, Eq. 103 is equivalent to

$$\Delta_M \equiv \sum_{n=M}^{\infty} (|\psi_n|^2 - |\psi'_n|^2) \leq 0, \quad \text{for } M = 0, 1, 2, \dots \quad (104)$$

Proving the minimum average photon-number property is now straightforward:

$$\begin{aligned} \langle \psi | \hat{N} | \psi \rangle - \langle \psi' | \hat{N} | \psi' \rangle &= \sum_{n=0}^{\infty} n (|\psi_n|^2 - |\psi'_n|^2) \\ &= \sum_{M=1}^{\infty} \Delta_M \leq 0. \end{aligned} \quad (105)$$

Thus, Eqs. 97 and 98 provide the means for choosing a state of minimum average energy and prescribed phase-measurement PDF.

## 8. PHASE COMMUNICATION

Sections 1–7 constitute an abridged version of Shapiro and Shepard.<sup>4</sup> That paper presents additional details regarding the state that achieves  $\delta\phi \sim 1/N^2$ , as well as substantial material on new classes of quantum states—coherent phase states, squeezed phase states, rational phase states—that are closely associated with the SG-POM. Furthermore, it proves that the Pegg-Barnett Hermitian phase operator<sup>16, 17</sup>—which exists on a truncated state space and provides phase-measurement statistics on the full state space through a limiting procedure—is included within the SG-POM formalism, i.e., these two schema produce *identical* phase measurement statistics for *all* quantum states. Neither of these topics will be considered herein. Instead, we shall move away from the *single-mode* case and develop new results for *two-mode* quantum phase measurement. Our objective will be to exploit the  $\hat{Y}$ -measurement—developed en route to the commuting observables form of the SG-POM—when the signal and apparatus modes are quantum-mechanically correlated and a phase-conjugate modulation is applied to them.

Consider the phase-conjugate quantum measurement setup shown in Fig. 1. This is a phase-conjugate system because whatever c-number phase shift  $\Phi$  is applied to the signal mode, leading to the annihilation operator transformation

$$\hat{a}_S^{IN} \longrightarrow e^{i\Phi} \hat{a}_S^{IN}, \quad (106)$$

the *conjugate* phase shift,  $-\Phi$  is applied to the apparatus mode, viz.

$$\hat{a}_A^{IN} \longrightarrow e^{-i\Phi} \hat{a}_A^{IN}, \quad (107)$$

cf. Eq. 30. If we take the signal and apparatus modes to be the appropriate linear polarizations, a transverse electro-optic modulator can be used to induce the necessary conjugate phase shifts.<sup>18</sup> Phase-conjugate shifts also appear, prototypically, in gravity-wave detecting interferometers. In fact, there are fundamental advantages to operating a phase-sensing interferometer in phase-conjugate fashion.<sup>19</sup> Our work does *not* depend explicitly on the means by which this modulation is accomplished. Its principal

motivation is to circumvent the Paley-Wiener restriction that encumbers phase-measurement PDF's for single-mode fields. As we shall see, some startling new possibilities arise with two modes.

The Paley-Wiener condition applies to a single-mode phase PDF because this density is proportional to the squared magnitude of the Fourier transform,  $\{\Psi(e^{i\phi}) : -\pi < \phi \leq \pi\}$ , of a one-sided sequence,  $\{\psi_n : n = 0, 1, 2, \dots\}$ . We shall break out of this limit, in the two-mode case, through quantum correlation. On  $\mathcal{H}_S \otimes \mathcal{H}_A$ , the joint state space of the signal and apparatus input modes, we can construct *number-product vacuum* states of the form

$$\begin{aligned} |\psi\rangle_{IN} &= \psi_0|0\rangle_S|0\rangle_A + \sum_{n=1}^{\infty} (\psi_n|n\rangle_S|0\rangle_A \\ &\quad + \psi_{-n}|0\rangle_S|n\rangle_A), \end{aligned} \quad (108)$$

where

$$\sum_{n=-\infty}^{\infty} |\psi_n|^2 = 1. \quad (109)$$

The term number-product vacuum is appropriate for such  $|\psi\rangle_{IN}$  because, when the signal  $\times$  apparatus state is of this class, a measurement of the number-operator product— $\hat{N}_S \otimes \hat{N}_A$ —yields outcome zero with probability one. Thus, for  $|\psi\rangle_{IN}$  a number-product vacuum state, Eqs. 62 and 63 imply that measurement of  $\hat{Y}$  yields a classical phasor  $e^{i\phi}$ , with  $-\pi < \phi \leq \pi$ . Moreover,  $\phi$  in this case has PDF

$$\begin{aligned} p(\phi | \Phi) &= \frac{|\Psi(e^{i(\phi-\Phi)})|^2}{2\pi}, \\ &\text{for } -\pi < \phi, \Phi \leq \pi, \end{aligned} \quad (110)$$

in terms of

$$\begin{aligned} \Psi(e^{i\phi}) &\equiv \sum_{n=-\infty}^{\infty} \psi_n e^{-in\phi}, \\ &\text{for } -\pi < \phi \leq \pi. \end{aligned} \quad (111)$$

Note that  $\{\psi_n : |n| = 0, 1, 2, \dots\}$  and  $\{\Psi(e^{i\phi}) : -\pi < \phi \leq \pi\}$  are *not* the number and phase representations, respectively, of *any* single-mode field state. They are, however, the number and phase representations, respectively, for a two-mode, number-product vacuum

state. The  $\{\psi_n, \Psi(e^{i\phi})\}$  notation is convenient because, as shown by Eq. 111 and its inverse,

$$\psi_n = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \Psi(e^{i\phi}) e^{in\phi},$$

for  $|n| = 0, 1, 2, \dots$ , (112)

these functions are a Fourier transform pair. More importantly, this notation makes clear the fact that number-ket causality does *not* restrict the possible two-mode phase PDF's. In particular, there are number-product vacuum states that satisfy

$$|\Psi(e^{i\phi})| = 0, \text{ for } |\phi| \geq \phi_c,$$

with  $\phi_c < \pi$ . (113)

i.e., two-mode phase PDF's can be confined to subintervals of  $(-\pi, \pi]$ , a situation that is forbidden to the single-mode case, cf. Eq. 96. This possibility is of great significance for phase-based digital communication and phase-based precision measurement, as we shall see.

To cast the Fig. 1 structure into a digital communication mold, let us assume that transverse electro-optic modulation is used to transmit a randomly-selected digit  $k$ , satisfying  $1 \leq k \leq K$ , by using  $\Phi = \Phi_k$ , where

$$\Phi_k \equiv -\pi + \frac{(2k-1)\pi}{K}. \quad (114)$$

Our objective is to make a minimum error probability decision as to which  $\Phi_k$  was sent, based on the result of the  $\hat{Y}$ -measurement when the signal  $\times$  apparatus state is a number-product vacuum, characterized by  $\{\psi_n, \Psi(e^{i\phi})\}$ .

Hall and Fuss have considered the *single-mode* version of this  $K$ -ary digital communication problem<sup>20</sup>. They optimized a single-mode state—in conjunction with the SG-POM—to obtain a phase-based quantum communication setup whose error probability vs. average photon number is significantly better than that for optical heterodyne detection. Hall and Fuss found a *nonzero* error probability at *finite* average photon number,  $N$ , which approached zero as  $N \rightarrow \infty$ . Surprisingly, in our two-mode problem, *zero* error

probability can be achieved at *finite* root-mean-square (RMS) photon number.<sup>21</sup>

In order to achieve zero error probability in phase-conjugate quantum communication, we need only use a number-product vacuum state which enforces Eq. 96, with  $\phi_c \leq \pi/K$ . Under this condition, we have that the *observed* phase,  $\phi$ , satisfies

$$\Pr\left(|\phi - \Phi_k| < \frac{\pi}{K} \mid \Phi = \Phi_k\right) = 1. \quad (115)$$

We also have that  $|\Phi_k - \Phi_j| \geq 2\pi/K$ , for  $j \neq k$ . So, for any observed  $\phi$  in the interval  $(-\pi, \pi]$ , we know that

$$\Pr\left(\Phi = \Phi_j \mid |\phi - \Phi_k| < \frac{\pi}{K}\right) = 0,$$

for all  $j \neq k$ . (116)

This means we can *unambiguously* determine which digit was sent by choosing the index associated with the *unique*  $\Phi$ -value that is within  $\pi/K$  rad of the observed phase. Via this procedure we decode  $k$  from  $\phi$  with zero probability of being incorrect.

To make this technique for zero error probability communication more explicit, we shall introduce a specific input state which has the desired property. The number representation we shall presume is

$$\psi_n = \sqrt{\frac{2}{K}} \frac{1}{\left(1 + \frac{2|n|}{K}\right)} \times \frac{\sin\left[\frac{\pi}{2}\left(1 - \frac{2|n|}{K}\right)\right]}{\frac{\pi}{2}\left(1 - \frac{2|n|}{K}\right)},$$

for  $|n| = 0, 1, 2, \dots$ ,  
and  $K = 2, 3, 4, \dots$  (117)

The associated phase representation for this state is easily computed to be

$$\Psi(e^{i\phi}) = \begin{cases} \sqrt{2K} \cos\left(\frac{K\phi}{2}\right), & \text{for } |\phi| \leq \frac{\pi}{K} \\ 0, & \text{for } \frac{\pi}{K} < |\phi| \leq \pi. \end{cases} \quad (118)$$

In Fig. 2 we have plotted  $p(\phi \mid \Phi = 0)$  vs.  $\phi$  for this state when  $K = 4$ ; we see that

the nonzero support of this PDF is the interval  $(-\pi/4, \pi/4)$ . In Fig. 3 we indicate how this PDF leads to zero error probability phase-conjugate communication; this figure plots the four conditional PDF's,  $\{p(\phi | \Phi_k) : 1 \leq k \leq 4\}$ , which apply when  $K = 4$ . For any *observed*  $\phi$ -value we must have that  $p(\phi | \Phi) > 0$ , otherwise that  $\phi$ -value could not have occurred. Figure 3 shows that, for any given  $\phi$ , there is only *one* possible  $\Phi_k$ -value which satisfies the nonzero PDF requirement—zero error probability communication results from deciding that this value was the transmitted phase.

The next question to address is the photon number statistics associated with our phase-conjugate communication scheme. For the general number-product vacuum state we have that the *total*—signal plus apparatus—photon number measurement has the following probability distribution,

$$\Pr(\hat{N}_S + \hat{N}_A = n) = \begin{cases} |\psi_0|^2, & \text{for } n = 0, \\ |\psi_{-n}|^2 + |\psi_n|^2, & \text{for } n = 1, 2, 3, \dots \end{cases} \quad (119)$$

For the particular state given by Eq. 117, it is then a simple matter to show that

$$\begin{aligned} \langle \hat{N}_S + \hat{N}_A \rangle &< \sqrt{\langle (\hat{N}_S + \hat{N}_A)^2 \rangle} \\ &= \frac{K}{2}. \end{aligned} \quad (120)$$

In other words, we can achieve zero error probability  $K$ -ary phase-conjugate quantum communication with an RMS *total* photon number of  $K/2$ . Figure 4 is a plot of Eq. 119 for the  $K = 4$  case.

The preceding quantum communication result is, of course, idealized. We have presumed a state generator—to produce a specific number-product vacuum state—that as yet has no explicit realization. Likewise, our scheme uses the  $\hat{Y}$ -measurement; again, no explicit realization is yet available. At least we can say that electro-optic modulation will impress the phase information on the input state, once that state can be produced. On the other hand, we have implicitly assumed lossless transmission; inclusion of loss will inevitably lead to nonzero error probability.

Our main purpose in going to the two-mode construct was to develop *potential* quantum-phase measurement schemes that promise substantial benefits, i.e., benefits that warrant the effort to bring them to fruition. This motivation is very much in line with the starting point for squeezed-state research.<sup>22</sup> In this regard, it is instructive to compare our phase-based scheme for zero error probability quantum communication with a more well-known approach based on number kets. For a single-mode field with annihilation operator  $\hat{a}$ , lossless transmission of one of the number kets  $\{|k-1\rangle : 1 \leq k \leq K\}$  followed by ideal direct detection, viz. the  $\hat{N} = \hat{a}^\dagger \hat{a}$  measurement, also yields  $K$ -ary digital communication without error. For  $k$  equally likely to be any digit between 1 and  $K$ , the *average*-energy efficiency of such a single-mode, number-ket system is roughly the same as that of our two-mode, phase-conjugate system, i.e., both need slightly less than  $K/2$  photons on average. The number-ket system has the advantage that its state generator may be approximated via feed-forward control using photon-twin beams, and its measurement only requires a high quantum-efficiency photon counter. Also, the number-ket approach uses less bandwidth; only one mode is needed. Alternatively, number-ket direct detection on a two-mode field can be used for error-free  $K$ -ary communication at significantly less than  $K/2$  photons on average. However, if we shift our attention from phase-based *communication*, to phase-based *precision measurements*, the Fig. 1 arrangement has a capability that number kets cannot match—phase sensing with controlled precision.

Suppose that we use the Fig. 1 arrangement for phase-conjugate precision measurement. Specifically, let us use the number-product vacuum state Eq. 117 in conjunction with a phase-conjugate interferometer (see, e.g., Bondurant and Shapiro<sup>19</sup>) and the  $\hat{Y}$  measurement. Now, the phase shift  $\Phi$  takes on any value from the continuum  $(-\pi, \pi]$ . Nevertheless, except for  $2\pi$ -modularity effects which come into play when  $\Phi$  is within  $\pi/K$  of  $\pm\pi$ , the observed phase will lie within  $\pi/K$  rad of the true phase *with probability one*. Thus, using less than  $K/2$  photons on average, we can guarantee a phase measure-

ment which is within  $\pi/K$  rad of the exact value. In other words, unlike more conventional schemes—which only ensure an acceptable RMS phase-estimation error—our phase-conjugate interferometer provides *exact* phase determination to a *prescribed* number of decimal places.

## 9. REFERENCES

1. P.A.M. Dirac, Proc. R. Soc. Lond. Ser. A **114**, 243 (1927).
2. L. Susskind and J. Glogower, Physics **1**, 49 (1964).
3. J.H. Shapiro, S.R. Shepard, and N.C. Wong, Phys. Rev. Lett. **62**, 2377 (1989).
4. J.H. Shapiro and S.R. Shepard, Phys. Rev. A **43**, 3795 (1991).
5. W.H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill, New York, 1964).
6. R.J. Glauber, Phys. Rev. **131**, 2766 (1963).
7. C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).
8. H.P. Yuen and J.H. Shapiro, IEEE Trans. Inform. Theory **IT-26**, 78 (1980).
9. J.H. Shapiro and S.S. Wagner, IEEE J. Quantum Electron. **QE-20**, 803 (1984).
10. J.H. Shapiro, IEEE J. Quantum Electron. **QE-21**, 237 (1985).
11. J.J. Stiffler, *Theory of Synchronous Communications* (Prentice Hall, Englewood Cliffs, 1971).
12. S.L. Braunstein, C.M. Caves, and A.S. Lane, "Maximum Likelihood Statistics of Multiple Quantum Phase Measurements," *Technical Digest Opt. Soc. Am. 1990 Annual Meeting*, (Opt. Soc. Am., Washington, D.C., 1990) p. 114.
13. P. Carruthers and M.M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).
14. A.V. Oppenheim and R.W. Schaffer, *Digital Signal Processing* (Prentice-Hall, Englewood Cliffs, 1975).
15. A. Papoulis, *Signal Analysis* (McGraw-Hill, New York, 1977).
16. D.T. Pegg and S.M. Barnett, Phys. Rev. A **39**, 1665 (1989).
17. S.M. Barnett and D.T. Pegg, J. Mod. Opt. **36**, 7 (1989).
18. A. Yariv, *Optical Electronics, Third Edition* (Holt, Rinehart, Winston, New York, 1985).
19. R.S. Bondurant and J.H. Shapiro, Phys. Rev. D **30**, 2548 (1984).
20. M.J.W. Hall and I.G. Fuss, "Quantum Phase Detection and Digital Communication," Quant. Opt. to appear.
21. J.H. Shapiro, "Phase Conjugate Quantum Communication with Zero Error Probability at Finite Average Photon Number," in preparation.
22. H.P. Yuen, Phys. Rev. A **13**, 2226 (1976).

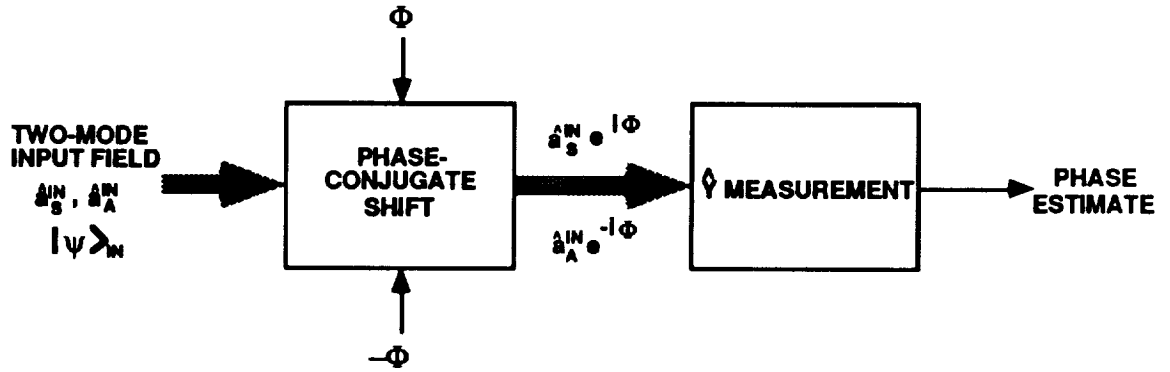


Figure 1: Phase-conjugate quantum communication system.

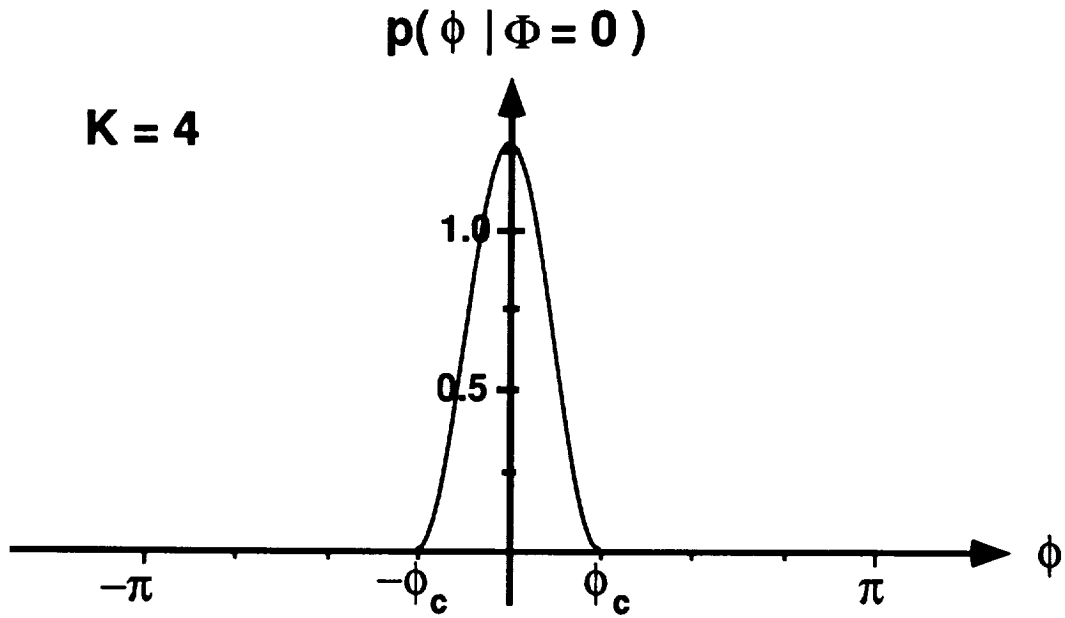


Figure 2: Conditional phase-measurement PDF, given  $\Phi = 0$ , for the state Eq. 118 when  $K = 4$ .



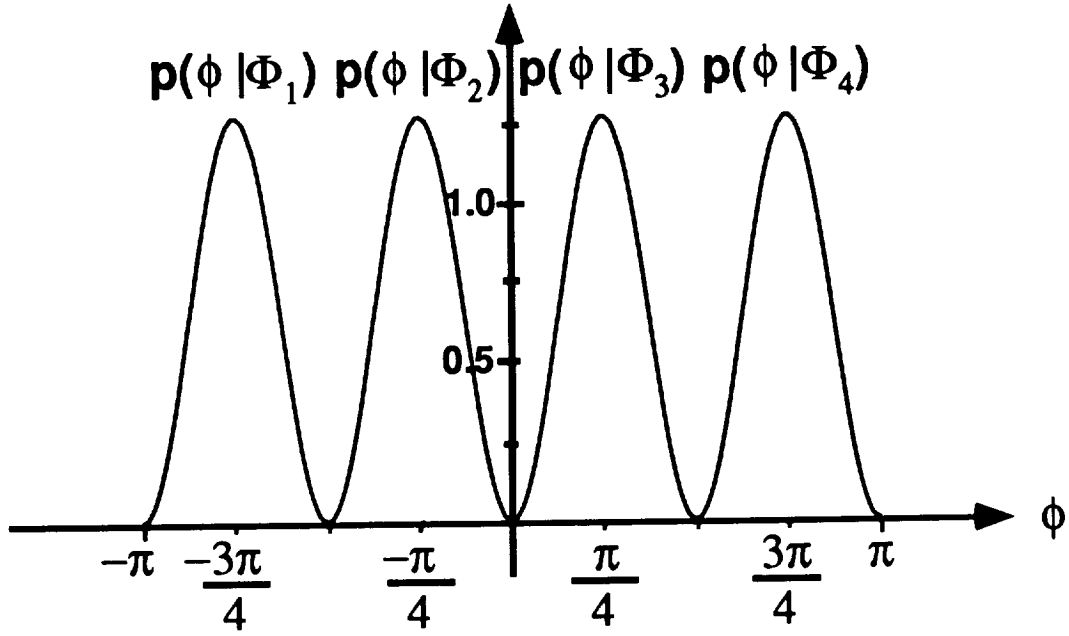


Figure 3: Conditional phase-measurement PDF's for the state Eq. 118 when  $K = 4$ .

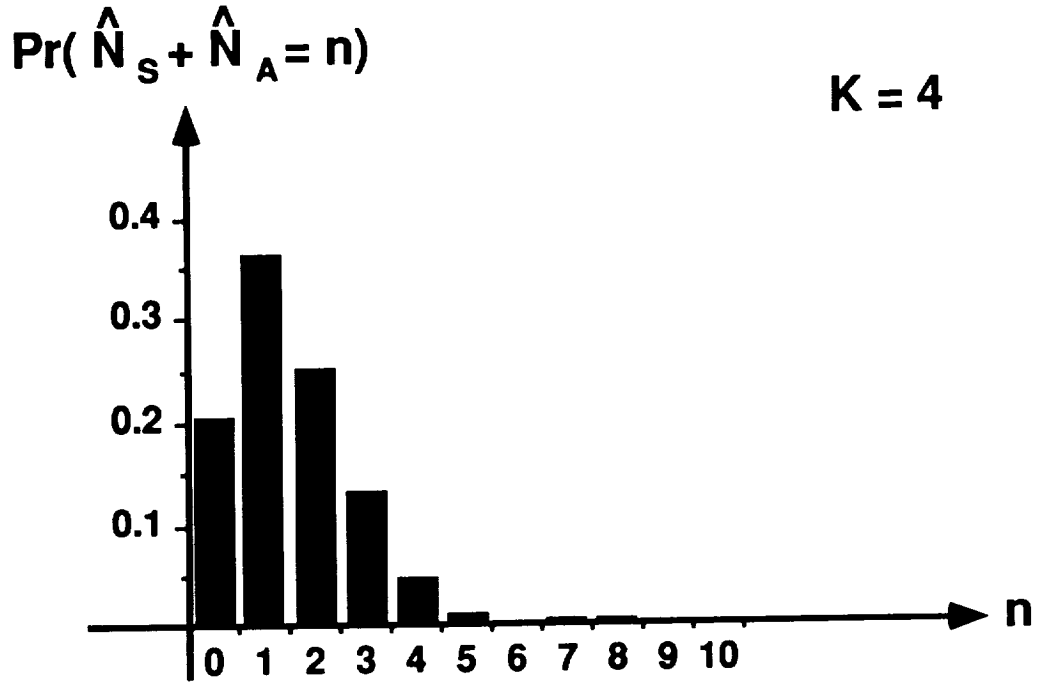


Figure 4: Signal-plus-apparatus number distribution for the state Eq. 117 when  $K = 4$ .